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Multiplicity-free tensor products of irreducible representations of the exceptional Lie groups

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Abstract
For each of the exceptional Lie groups, a complete determination is given of those pairs of finite-dimensional irreducible representations whose tensor products (or squares) may be resolved into irreducible representations that are multiplicity free, i.e. such that no irreducible representation occurs in the decomposition of the tensor product more than once. Explicit formulae are presented for the decomposition of all those tensor products that are multiplicity free, many of which exhibit a stability property.

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1. Introduction

The classification of the complex semisimple Lie algebras was completed by the 25-year-old Elie Cartan in his thesis [1] of 1894. Four great classes of complex simple Lie algebras were identified and designated by Cartan as A_k, B_k, C_k and D_k. These Lie algebras are associated with the classical Lie groups SU_{k+1}, SO_{2k+1}, Sp_{2k} and SO_{2k}, respectively. In addition to the four classes of classical Lie algebras, Cartan identified five exceptional Lie algebras which he designated as G_2, F_4, E_6, E_7 and E_8, where the subscripted integers are the ranks of the respective algebras.

Starting with Racah's [2] use of G_2 in his analysis of the complex spectra of f-electron configurations the exceptional Lie groups have become of increasing interest to physicists. In atomic physics Wadzinski [3], and later Judd [4], made use of the group F_4, while in the interacting boson model of nuclei [5] use was made of E_6. Particle physicists developed a considerable interest in the possibilities of the exceptional Lie groups in the formulation of grand unified theories of the fundamental forces as typified by the review of Gell-Mann *et al* [6] in 1978. A storm of interest in the exceptional Lie group E_8 was created in 1984 by Green and Schwarz's dramatic development of superstring theories [7]. This storm remains unabated [8].

The above developments have stimulated much work by both mathematicians and physicists on the basic properties of the exceptional Lie groups and their representations, in particular the systematic labelling of their finite-dimensional irreducible representations (irreps) [9–15], the explicit evaluation of their characters expressed in terms of weight multiplicities [9–11, 16, 17], the determination of group–subgroup decompositions in the form of branching rules [9, 11, 18] and the decomposition of tensor products of irreps [9, 12, 13].

In this paper we give a complete determination, for each of the exceptional Lie groups, of those irreps whose tensor products (or squares) are multiplicity free, i.e. in the resolution of the tensor product as a direct sum of irreps no irrep occurs more than once. Our interest in this problem was stimulated by Stembridge's [19] recent classification first of all multiplicity-free products of Schur functions and then of all multiplicity-free products of characters of $SL(n, \mathbb{C})$. The motivation for examining multiplicity-free tensor products is based on the fact that their centralizer algebras are Abelian and their decompositions are in some sense canonical [20]. In corresponding physical applications their particular merit is that they are free of the complexities that arise from the missing label problem [21, 22].

We start in section 2 by defining tensor product multiplicities and linking them to weight multiplicities. In particular we state a number of known results that put constraints on tensor product multiplicities. These constraints form the basis of our analysis in later sections of tensor product multiplicities of irreps of the exceptional Lie groups. In section 3 we review the labelling of the finite-dimensional irreps of the exceptional groups both in terms of Dynkin labels [14, 15] and partitions [9–12], noting the relationship between the two schemes. The defining, adjoint, fundamental and what we call scaled fundamental irreps are all identified, as well as the breadth and Dynkin weight of an arbitrary irrep, concepts that are of particular value in what follows.

In section 4 we present an analysis, for the exceptional Lie groups, of tensor products of the defining and the adjoint irreps with an arbitrary irrep, identifying those that are multiplicity free and giving explicit formulae for their decomposition. This is followed in sections 5–10 by consideration of a sequence of increasingly general cases that allow us to systematically narrow down the search for those tensor products that may be multiplicity free. The first three cases are concerned with irreps of breadth one: first tensor products of pairs of fundamental irreps, then tensor products of fundamental irreps with scaled fundamental irreps and finally tensor products of pairs of scaled fundamental irreps. The last three cases involve irreps of breadth greater than one: first in tensor products with a fundamental irrep, then with a scaled fundamental and finally with another irrep having breadth greater than one. At each step some new multiplicity-free products are identified until the final step is reached where no new multiplicity-free products exist. Having determined all the possible multiplicity-free tensor products, the entire collection of such products is collected together by way of summary in section 11.

2. Tensor product multiplicities

Each finite-dimensional irreducible representation, V^λ, of a complex simple Lie algebra, $\boldsymbol{g}$, or the corresponding Lie group, G, may be specified, up to equivalence, by means of its highest weight λ, with $\lambda \in P^+$, the set of dominant integral weights of $\boldsymbol{g}$. The tensor product of any pair of such irreps, V^μ and V^ν with $\mu, \nu \in P^+$, is fully reducible, and its decomposition takes the form

$$V^\mu \otimes V^\nu = \sum_{\lambda \in P^+} c(\lambda; \mu, \nu)\ V^\lambda, \tag{2.1}$$

where the sum is a direct sum, and the coefficients $c(\lambda; \mu, \nu)$ are the tensor product multiplicities giving the number of times the irrep V^λ appears as a summand in this decomposition. A tensor product of the type (2.1) is said to be multiplicity free if

$$c(\lambda; \mu, \nu) \leqslant 1 \qquad \text{for all } \lambda. \tag{2.2}$$

The tensor product multiplicities satisfy the symmetry conditions

$$c(\lambda; \mu, \nu) = c(\lambda; \nu, \mu) = c(\overline{\lambda}; \overline{\mu}, \overline{\nu}) = c(\overline{\mu}; \overline{\lambda}, \nu), \tag{2.3}$$

where $\overline{\lambda}$ is the highest weight of the irrep $V^{\overline{\lambda}}$ that is contragredient to V^λ.

There are at least two explicit formulae for the tensor product multiplicities. The first of these can be readily derived through the use of characters. The character of the irrep V^λ is defined by

$$\mathrm{ch}\, V^\lambda = \sum_{\gamma \in P(\lambda)} m_\gamma^\lambda \mathrm{e}^\gamma, \tag{2.4}$$

where $P(\lambda)$ is the set of all weights of the irrep V^λ, and m_γ^λ is the multiplicity of the weight γ in the irrep V^λ, that is the dimension, $\dim V_\gamma^\lambda$, of the subspace of V^λ spanned by vectors having weight γ. The irrep V^λ is said to be multiplicity free if

$$m_\gamma^\lambda = \dim V_\gamma^\lambda = 1 \qquad \text{for all } \gamma \in P(\lambda). \tag{2.5}$$

Each simple Lie algebra $\boldsymbol{g}$ of rank k possesses a set of fundamental irreps V^{ω_i} with highest weights ω_i for $i = 1, 2, \ldots, k$. The Weyl vector ρ, which is half the sum of the positive roots of $\boldsymbol{g}$, may be equally well defined by $\rho = \sum_{i=1}^k \omega_i$. In terms of this vector, Weyl's character formula [23, 24] can be expressed as

$$\mathrm{ch}\, V^\lambda = \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\lambda+\rho)} \Big/ \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\rho)}, \tag{2.6}$$

where W is the Weyl group of $\boldsymbol{g}$, and $\epsilon(w) = (-1)^{\ell(w)}$ where $\ell(w)$ is the length of w when expressed as a product of the reflections that generate W.

In this finite-dimensional fully reducible context the irreps V^λ and the decomposition of their tensor products are determined up to equivalence by their characters $\mathrm{ch}\, V^\lambda$. It follows from (2.4) and (2.6), and the invariance of weight multiplicities under the action of the Weyl group, that [25–27]

$$\begin{aligned} \mathrm{ch}\, V^\mu \times \mathrm{ch}\, V^\nu &= \sum_{\gamma \in P(\mu)} m_\gamma^\mu \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\nu+\gamma+\rho)} \Big/ \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\rho)} \\ &= \sum_{\gamma \in P(\mu)} m_\gamma^\mu\, \mathrm{ch}\, V^{\nu+\gamma} = \sum_{\lambda \in P^+} c(\lambda; \mu, \nu)\, \mathrm{ch}\, V^\lambda. \end{aligned} \tag{2.7}$$

Here $\nu + \gamma$ is not necessarily a dominant integral, but $\mathrm{ch}\, V^{\nu+\gamma}$ is either zero, by virtue of $\nu + \gamma + \rho$ lying on a Weyl reflection hyperplane in weight space, or is equal to $\epsilon(w)\mathrm{ch}\, V^\lambda$ where $\lambda \in P^+$ is obtained from $\nu + \gamma$ by the dot action of the Weyl group, that is $w \cdot (\nu + \gamma) = w(\nu + \gamma + \rho) - \rho = \lambda$ for some $w \in W$. From this follows our first formula for tensor product multiplicities [25–27]:

Theorem 1. *For all* $\lambda, \mu, \nu \in P^+$

$$c(\lambda; \mu, \nu) = \sum_{\substack{\gamma \in P(\mu) \\ w \cdot (\nu+\gamma) = \lambda \in P^+}} \epsilon(w) m_\gamma^\mu = \sum_{w \in W} \epsilon(w) m^\mu_{w \cdot \lambda - \nu}. \tag{2.8}$$

To present the second formula for these tensor product multiplicities we require some additional notation. Let the Chevalley generators of the simple Lie algebra $\boldsymbol{g}$ of rank k under consideration be $\{h_i, e_i, f_i \mid i = 1, 2, \ldots, k\}$. Then the simple module corresponding to the irrep V^λ can be realized through the action of these generators on a highest-weight vector v_λ. This action is such that

$$h_i v_\lambda = \lambda(h_i) v_\lambda, \quad e_i v_\lambda = 0 \quad \text{and} \quad f_i^{\lambda(h_i)+1} v_\lambda = 0 \qquad \text{for } i = 1, 2, \ldots, k, \tag{2.9}$$

where the non-negative integers $\lambda(h_i)$ are just the components of λ in the basis of weight space afforded by the fundamental weights ω_i, that is $\lambda = \sum_{i=1}^{k} \lambda(h_i)\omega_i$. From this realization of irreps a second formula for the tensor product multiplicities has been derived [28, 29]

Theorem 2. *For all* $\lambda, \mu, \nu \in P^+$

$$c(\lambda; \mu, \nu) = \dim(\{v \in V^\mu_{\lambda-\nu} \mid e_i^{\nu(h_i)+1} v = 0, i = 1, 2, \ldots, k\}). \tag{2.10}$$

It follows from both theorems 1 and 2 that we have [25–28]:

Corollary 1. *For all* $\lambda, \mu, \nu \in P^+$

$$c(\lambda; \mu, \nu) = m^\mu_{\lambda-\nu} \tag{2.11}$$

if either $\nu + \gamma \in P^+$ *for all* $\gamma \in P(\mu)$ *or* $\nu(h_i) + 1 \geqslant \dim V^\mu$ *for all* $i = 1, 2, \ldots, k$, *where* $\dim V^\mu$ *is the dimension of the irrep* V^μ.

Failing this, since $\dim V^\mu_{\lambda-\nu} = m^\mu_{\lambda-\nu}$, it follows immediately from theorem 2 that we have [29]:

Corollary 2. *For all* $\lambda, \mu, \nu \in P^+$

$$c(\lambda; \mu, \nu) \leqslant m^\mu_{\lambda-\nu}. \tag{2.12}$$

In addition, as pointed out by Stembridge [19], it also follows from theorem 2 that we have:

Corollary 3. *For all* $\lambda, \mu, \nu \in P^+$

$$c(\lambda + \omega_i; \mu, \nu + \omega_i) \geqslant c(\lambda; \mu, \nu) \qquad \text{for } i = 1, 2, \ldots, k. \tag{2.13}$$

Proof. Setting $\sigma = \nu + \omega_i$ we have $\sigma(h_i) = \nu(h_i) + 1$. If $e_i^{\nu(h_i)+1} v = 0$ it then follows that $e_i^{\sigma(h_i)+1} v = 0$. Since $(\lambda + \omega_i) - (\nu + \omega_i) = \lambda - \nu$, the result (2.13) follows directly from (2.10). □

In (2.13) it is possible to establish some conditions under which a certain stability sets in as a result of the inequality becoming an equality. To be precise we have:

Corollary 4. *Let* i *be fixed. Then for all* $\lambda, \mu, \nu \in P^+$ *such that* $\nu(h_i) = n$ *with* $n \geqslant n_s$, *for some sufficiently large but finite positive integer* n_s,

$$c(\lambda + \omega_i; \mu, \nu + \omega_i) = c(\lambda; \mu, \nu). \tag{2.14}$$

Proof. It is first necessary to show that n_s exists. Since the irrep V^μ is finite dimensional, it follows that for each v in the corresponding module there must exist a unique smallest positive finite integer $n_i(v)$ such that $e_i^{n_i(v)-1} v \neq 0$ and $e_i^{n_i(v)} v = 0$. Now set

$$n_s = \max\{n_i(v) \mid v \in V^\mu_{\lambda-\nu}\}. \tag{2.15}$$

Then provided that $\nu(h_i) = n \geqslant n_s$ we have $e_i^{\nu(h_i)+1} v = 0$ for all $v \in V^\mu_{\lambda-\nu}$. Furthermore, setting $\sigma = \nu + \omega_i$, we have $\sigma(h_i) = \nu(h_i) + 1$ and $\sigma(h_j) = \nu(h_j)$ for $j \neq i$. It follows that $e_i^{\sigma(h_i)+1} v = 0$ and $e_j^{\sigma(h_j)+1} v = e_j^{\nu(h_j)+1} v$ for all $j \neq i$ and all $v \in V^\mu_{\lambda-\nu}$. The required result then follows from theorem 2. □

Table 1. Standard labels for irreps of the exceptional groups of rank k, where $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition into no more than k non-vanishing parts, so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$ with λ_i a non-negative integer for $i = 1, 2, \ldots, k$.

Group	Label	Constraints
G_2	$(\lambda) = (\lambda_1, \lambda_2)$	$\lambda_1 \geqslant 2\lambda_2$
F_4	$(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	$\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4$
	$(\Delta; \lambda) = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \lambda_3 + \frac{1}{2}, \lambda_4 + \frac{1}{2})$	$\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4$
E_6	$(\lambda) = (\lambda_1 : \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$	$\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6$
E_7	$(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$	$\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7$
E_8	$(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$	$\lambda_1 \geqslant 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8$

3. Labelling of irreps of the exceptional Lie groups

The irrep V^λ of an exceptional Lie group of rank k may be labelled up to equivalence by specifying its highest weight λ in the fundamental weight basis through the relation $\lambda = \sum_{i=1}^{k} a_i \omega_i$, where as we have indicated $a_i = \lambda(h_i)$. The corresponding vector of coefficients, which we write as $((a)) = ((a_1, a_2, \ldots, a_k))$, is the label for the irrep that is conventionally associated with the Dynkin diagram appropriate to the group in question [14,15]. The condition that the irrep be finite dimensional is just the condition that a_i be a non-negative integer for all $i = 1, 2, \ldots, k$.

Alternatively, though equivalently, natural labelling schemes [9–11] each based upon that of a maximal classical subgroup of the exceptional Lie group of the same rank have been developed. In this case the standard labels are identical with those of the chosen subgroup, but subject to certain constraints. The relevant subgroups used herein are

$$\begin{aligned} &G_2 \supset SU_3 \\ &F_4 \supset SO_9 \\ &E_6 \supset SU_2 \times SU_6 \\ &E_7 \supset SU_8 \\ &E_8 \supset SU_9. \end{aligned} \tag{3.1}$$

The standard labels involve constrained partitions and are thus often referred to as partition labels. The standard labels for the exceptional Lie groups based upon the above subgroups have been given by Black *et al* [13] and are reproduced in table 1.

The correspondence between the Dynkin labels $((a))$ and the standard labels (λ) has been described elsewhere [10] and is reproduced in table 2. In the case of F_4 the labels (λ) and $(\Delta; \lambda)$ distinguish between integral and half-integral weights with Δ signifying the weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, while for E_6 the colon : is introduced to separate those parts of the weight associated with the SU_2 and SU_6 weight spaces.

The use of Dynkin labels is particularly helpful in defining two parameters that play a crucial role in what follows, namely the *breadth*, $b(\lambda)$, and the *Dynkin weight*, $w(\lambda)$ of a dominant integral highest weight λ. These are defined for $\lambda = ((a_1, a_2, \ldots, a_k))$ by

$$b(\lambda) = \#\{a_i > 0 \mid i = 1, 2, \ldots, k\} \qquad \text{and} \qquad w(\lambda) = \sum_{i=1}^{k} a_i, \tag{3.2}$$

where $\#\{\cdots\}$ indicates the number of elements in the set $\{\cdots\}$. It should be noted that the fundamental irreps have highest weights ω_i that are characterized by the fact that they have breadth $b(\omega_i) = 1$ and Dynkin weight $w(\omega_i) = 1$ for all $i = 1, 2, \ldots, k$.

Table 2. Correspondence between the Dynkin labels and the standard labels for the irreps of the exceptional Lie groups.

Group	Dynkin label $((a))$	Standard label (λ)
G_2	$a_1 = \lambda_2$	$\lambda_1 = 2a_1 + a_2$
	$a_2 = \lambda_1 - 2\lambda_2$	$\lambda_2 = a_1$
F_4	$a_1 = \lambda_2 - \lambda_3$	$\lambda_1 = a_1 + 2a_2 + \frac{3}{2}a_3 + a_4$
	$a_2 = \lambda_3 - \lambda_4$	$\lambda_2 = a_1 + a_2 + \frac{1}{2}a_3$
	$a_3 = 2\lambda_4$	$\lambda_3 = a_2 + \frac{1}{2}a_3$
	$a_4 = \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$	$\lambda_4 = \frac{1}{2}a_3$
E_6	$a_1 = \lambda_2 - \lambda_3$	$\lambda_1 = a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6$
	$a_2 = \lambda_3 - \lambda_4$	$\lambda_2 = a_1 + a_2 + a_3 + a_4 + a_5$
	$a_3 = \lambda_4 - \lambda_5$	$\lambda_3 = a_2 + a_3 + a_4 + a_5$
	$a_4 = \lambda_5 - \lambda_6$	$\lambda_4 = a_3 + a_4 + a_5$
	$a_5 = \lambda_6$	$\lambda_5 = a_4 + a_5$
	$a_6 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6)$	$\lambda_6 = a_5$
E_7	$a_1 = \lambda_7$	$\lambda_1 = 2a_1 + 3a_2 + 4a_3 + 3a_4 + 2a_5 + a_6 + 2a_7$
	$a_2 = \lambda_6 - \lambda_7$	$\lambda_2 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$
	$a_3 = \lambda_5 - \lambda_6$	$\lambda_3 = a_1 + a_2 + a_3 + a_4 + a_5$
	$a_4 = \lambda_4 - \lambda_5$	$\lambda_4 = a_1 + a_2 + a_3 + a_4$
	$a_5 = \lambda_3 - \lambda_4$	$\lambda_5 = a_1 + a_2 + a_3$
	$a_6 = \lambda_2 - \lambda_3$	$\lambda_6 = a_1 + a_2$
	$a_7 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 + \lambda_6 + \lambda_7)$	$\lambda_7 = a_1$
E_8	$a_1 = \lambda_8$	$\lambda_1 = 2a_1 + 3a_2 + 4a_3 + 5a_4 + 6a_5 + 4a_6 + 2a_7 + 3a_8$
	$a_2 = \lambda_7 - \lambda_8$	$\lambda_2 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$
	$a_3 = \lambda_6 - \lambda_7$	$\lambda_3 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$
	$a_4 = \lambda_5 - \lambda_6$	$\lambda_4 = a_1 + a_2 + a_3 + a_4 + a_5$
	$a_5 = \lambda_4 - \lambda_5$	$\lambda_5 = a_1 + a_2 + a_3 + a_4$
	$a_6 = \lambda_3 - \lambda_4$	$\lambda_6 = a_1 + a_2 + a_3$
	$a_7 = \lambda_2 - \lambda_3$	$\lambda_7 = a_1 + a_2$
	$a_8 = \frac{1}{3}(\lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8)$	$\lambda_8 = a_1$

So as to remove any ambiguities in the numbering of the fundamental weights, the labels for the fundamental irreps and their dimensions are displayed in table 3, in which the highest weights of the defining (or lowest-dimensional non-trivial) irrep, ω, its contragredient $\overline{\omega}$ (if different) and the adjoint irrep θ have all been identified, for each exceptional Lie group[3].

On the other hand the great merit of what we have called the standard labels for irreps of the exceptional Lie groups, G, is that they coincide with the labels used for irreps of the maximal classical subgroups, H, of the same rank that are listed in (3.1). This allows us to write down and exploit the following formula for the evaluation of tensor products of irreps of the exceptional Lie groups [12, 13]:

$$\mathrm{ch}\, V_G^{\mu} \times \mathrm{ch}\, V_G^{\nu} = \sum_{\sigma,\tau\in P_H^+} b_{\sigma}^{\mu}\, c_H(\tau;\sigma,\nu+\delta)\, \mathrm{ch}\, V_G^{\tau-\delta} = \sum_{\lambda\in P_G^+} c_G(\lambda;\mu,\nu)\, \mathrm{ch}\, V_G^{\lambda}. \tag{3.3}$$

Here the summation is carried out over all σ and τ that are dominant integral weights of the classical subgroup H, and $\delta = \rho_G - \rho_H$ where ρ_G and ρ_H are the Weyl vectors of G and H, respectively. In this formula (3.3) the coefficients b_{σ}^{μ} are the branching rule multiplicities giving the number of times the irrep V_H^{σ} of the classical group H appears in the restriction from G to

[3] We take the opportunity to point out an error in table 1 of [17]. The Dynkin label for the adjoint representation (θ) of $SU(k+1) = A_k$ should have read $((a)) = ((100\cdots001))$.

Table 3. The fundamental irreps of the exceptional Lie groups; their numbering, labels and dimensions.

G	ω_i	$((a))$	(λ)	Dimension
G_2	$\omega_1 = \theta$	10	(21)	14
	$\omega_2 = \omega$	01	(1)	7
F_4	$\omega_1 = \theta$	1 000	(1^2)	52
	ω_2	0 100	(21^2)	1 274
	ω_3	0 010	$(\Delta\ 1)$	273
	$\omega_4 = \omega$	0 001	(1)	26
E_6	$\omega_1 = \omega$	100 000	(1 : 1)	27
	ω_2	010 000	$(2 : 1^2)$	351
	ω_3	001 000	$(3 : 1^3)$	2 925
	ω_4	000 100	$(2 : 1^4)$	351
	$\omega_5 = \overline{\omega}$	000 010	$(1 : 1^5)$	27
	$\omega_6 = \theta$	000 001	(2 : 0)	78
E_7	$\omega_1 = \theta$	1 000 000	(21^6)	133
	ω_2	0 100 000	(31^5)	8 645
	ω_3	0 010 000	(41^4)	365 750
	ω_4	0 001 000	(31^3)	27 664
	ω_5	0 000 100	(21^2)	1 539
	$\omega_6 = \omega$	0 000 010	(1^2)	56
	ω_7	0 000 001	(2)	912
E_8	$\omega_1 = \omega = \theta$	10 000 000	(21^7)	248
	ω_2	01 000 000	(31^6)	30 380
	ω_3	00 100 000	(41^5)	2 450 240
	ω_4	00 010 000	(51^4)	146 325 270
	ω_5	00 001 000	(61^3)	6899 079 264
	ω_6	00 000 100	(41^2)	6 696 000
	ω_7	00 000 010	(21)	3 875
	ω_8	00 000 001	(3)	147 250

H of the irrep V_G^μ of the exceptional Lie group G. The coefficients $c_H(\tau; \sigma, \nu + \delta)$ are tensor product multiplicities associated with the classical group H, and as such may be evaluated by standard methods. The only subtlety arising in the exploitation of this formula is that the last step leading to the evaluation of the tensor product coefficients $c_G(\lambda; \mu, \nu)$ of the exceptional Lie group G may require the use of the dot action of the Weyl group of G to convert any non-standard label $\tau - \delta \notin P_G^+$ to some standard label $\lambda \in P_G^+$. However, the requisite modification rules have all been identified previously [12, 13]. These include all the modification rules associated with the classical subgroup H, augmented by a single additional modification rule associated exclusively with the exceptional Lie group G through a violation of the constraints listed in table 1. For each exceptional Lie group G these take the form given in table 4.

The result of the application of these and other modification rules to a non-standard labelled irrep is to either produce a null result or a standard labelled irrep with a possible change of sign. Extensive examples of the application of modification rules have been given by Black *et al* [13]. Here, by way of example, we note that the G_2 irrep labels (32) and (64) are non-standard. Using the relevant formula of table 4 we find for $(\lambda) = (32)$ that $h = 0$ and hence $w \cdot (32) = (32)$ with $\epsilon(w) = -1$, from which it follows that $\mathrm{ch}\, V^{(32)} = -\mathrm{ch}\, V^{(32)} = 0$. On the other hand for $(\lambda) = (64)$ we find $h = 1$ and hence $w \cdot (64) = (63)$ with $\epsilon(w) = -1$, from which it follows that $\mathrm{ch}\, V^{(64)} = -\mathrm{ch}\, V^{(63)}$.

Table 4. Weight space modification rules of the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \to w \cdot \lambda = w(\lambda + \rho_G) - \rho_G$ with $w \in W_G \backslash W_H$ and $\epsilon(w) = -1$ to be used if $h \geqslant 0$.

G	h	$w \cdot \lambda$
G_2	$-(\lambda_1 - 2\lambda_2 + 1)$	$(\lambda_1, \lambda_2 - h)$
F_4	$-\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + 1)$	$(\lambda_1 + h, \lambda_2 - h, \lambda_3 - h, \lambda_4 - h)$
E_6	$-\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 + 2)$	$(\lambda_1 + h, \lambda_2 - h, \lambda_3 - h, \lambda_4 - h, \lambda_5, \lambda_6)$
E_7	$-\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 + \lambda_6 + \lambda_7 + 2)$	$(\lambda_1, \lambda_2 - h, \lambda_3 - h, \lambda_4 - h, \lambda_5 - h, \lambda_6, \lambda_7)$
E_8	$-\frac{1}{3}(\lambda_1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + 3)$	$(\lambda_1, \lambda_2 - h, \lambda_3 - h, \lambda_4 - h, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$

There is a sense in which (3.3) is an analogue of (2.7), with the role of the subgroup H in (3.3) being played by the subgroup $U(1)^{\otimes k}$ in (2.7). This becomes clear by noting that this latter subgroup is such that the branching rule multiplicities for the restriction from G to $U(1)^{\otimes k}$ are nothing other than the weight multiplicities of G. By the same token the analogue of theorem 1 takes the form:

Theorem 3.

$$c(\lambda; \mu, \nu) = \sum_{\substack{\sigma, \tau \in P_H^+ \\ w \cdot (\tau - \delta) = \lambda}} \epsilon(w)\, b_\sigma^\mu\, c_H(\tau; \sigma, \nu + \delta)$$
$$= \sum_{\substack{w \in W \\ \sigma, w \cdot \lambda + \delta \in P_H^+}} \epsilon(w)\, b_\sigma^\mu\, c_H(w \cdot \lambda + \delta; \sigma, \nu + \delta). \qquad (3.4)$$

In the calculations that follow it is the branching rule algorithm based on (3.3), or equivalently (3.4), that is used to evaluate tensor products of irreps of the exceptional Lie groups, rather than any weight multiplicity algorithm based on (2.7), or equivalently (2.8). The algorithm, complete with relevant branching rule data for the restriction from G to H, the tensor product procedures for H and all the necessary modification rules, has been implemented in SCHUR[4], and it is this programme that has been used to derive the results needed here.

We note that in exploiting (2.3) to reduce the number of independent tensor products that need to be evaluated, the irreps of the exceptional Lie groups are all self-contragredient save some of those of E_6. In the case of E_6 an irrep $V^{\overline{\lambda}}$ is contragredient to V^λ if their standard labels $(\overline{\lambda})$ and (λ) are related by [17]:

$$\overline{(\lambda_1 : \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)} = (\lambda_1 : \lambda_2, \lambda_2 - \lambda_6, \lambda_2 - \lambda_5, \lambda_2 - \lambda_4, \lambda_2 - \lambda_3), \qquad (3.5a)$$

or equivalently in terms of Dynkin labels

$$\overline{((a_1, a_2, a_3, a_4, a_5, a_6))} = ((a_5, a_4, a_3, a_2, a_1, a_6)). \qquad (3.5b)$$

Accordingly, an irrep V^λ of E_6 is self-contragredient if and only if

$$\lambda_2 = \lambda_3 + \lambda_6 = \lambda_4 + \lambda_5 \qquad \text{or equivalently} \qquad a_1 = a_5,\ a_2 = a_4. \qquad (3.5c)$$

For the sake of typographical simplicity, in what follows, we denote the irrep V^λ of highest weight λ by (λ) and the tensor product $V^\mu \otimes V^\nu$ by $(\mu) \times (\nu)$. Dynkin labels $((a))$ are usually abbreviated to just $a_1 a_2 \cdots a_k$. Since we are only considering irreps up to equivalence and all the tensor products are fully reducible, labels (λ) may also be interpreted as characters, ch V^λ for which the modification rules ch $V^\lambda = \epsilon(w)$ch $V^{w \cdot \lambda}$ may be written as $(\lambda) = \epsilon(w)(w \cdot \lambda)$ wherever they are required.

[4] SCHUR, an interactive program for calculating properties of Lie groups and symmetric functions, distributed by S Christensen. E-mail: steve@scm.vnet; http://scm.vnet/Christensen.html.

Table 5. The tensor product $(\omega) \times (\nu)$ for G_2.

$(1) \times (m, n)$	$= (m+1, n+1) + (m+1, n) + (m, n+1) + (m, n) + (m, n-1)$ $+ (m-1, n) + (m-1, n-1)$

Table 6. The tensor product $(\omega) \times (\nu)$ for E_6.

$(1:1) \times (m:n, p, q, r, s)$	$= (m+1:n+1, p, q, r, s) + (m+1:n, p+1, q, r, s) + (m+1:n, p, q+1, r, s)$ $+ (m+1:n, p, q, r+1, s) + (m+1:n, p, q, r, s+1)$ $+ (m+1:n-1, p-1, q-1, r-1, s-1) + (m:n+1, p+1, q+1, r+1, s)$ $+ (m:n+1, p+1, q+1, r, s+1) + (m:n+1, p+1, q, r+1, s+1)$ $+ (m:n+1, p, q+1, r+1, s+1) + (m:n, p+1, q+1, r+1, s+1)$ $+ (m:n, p, q, r-1, s-1) + (m:n, p, q-1, r, s-1)$ $+ (m:n, p, q-1, r-1, s)$ $+ (m:n, p-1, q, r, s-1) + (m:n, p-1, q, r-1, s)$ $+ (m:n, p-1, q-1, r, s)$ $+ (m:n-1, p, q, r, s-1) + (m:n-1, p, q, r-1, s)$ $+ (m:n-1, p, q-1, r, s)$ $+ (m:n-1, p-1, q, r, s) + (m-1:n+1, p, q, r, s)$ $+ (m-1:n, p+1, q, r, s)$ $+ (m-1:n, p, q+1, r, s) + (m-1:n, p, q, r+1, s)$ $+ (m-1:n, p, q, r, s+1)$ $+ (m-1:n-1, p-1, q-1, r-1, s-1)$

4. Tensor products with the defining and adjoint irreps

The cases of the tensor products $(\omega) \times (\nu)$ and $(\theta) \times (\nu)$ have been largely covered in two earlier works [12, 17].

All the defining irreps (ω) are weight multiplicity free except for those of F_4 and E_8. It follows from (2.12) that, with those exceptions,

$$G_2,\ E_6,\ E_7 : c(\lambda; \omega, \nu) \in \{0, 1\} \qquad \text{for all } (\nu), \tag{4.1a}$$

i.e. the tensor products $(\omega) \times (\nu)$ are multiplicity free for all (ν). In fact there only exists one other weight multiplicity-free irrep, namely the contragredient $(\overline{\omega})$ of the defining irrep of E_6. It follows that in addition to (4.1a) we have

$$E_6 : c(\lambda; \overline{\omega}, \nu) \in \{0, 1\} \qquad \text{for all } (\nu), \tag{4.1b}$$

i.e. the tensor products $(\overline{\omega}) \times (\nu)$ of E_6 are also multiplicity free for all (ν).

By using (2.7) we can be more precise, the tensor product decompositions take the forms given in tables 5–7.

By virtue of (2.3) the tensor product decomposition of $(\overline{\omega}) \times (\overline{\nu})$ for E_6 may be obtained from that for $(\omega) \times (\nu)$ given in table 6 by taking the contragredient of every term by means of (3.5a). This suffices to deal with the $(\overline{\omega})$ case since $(\overline{\nu})$ varies over all irreps just as (ν) does.

The decompositions given in tables 5–8 apply to all possible irreps (ν) of the relevant exceptional group G. However, for particular values of the parameters $m, n, \ldots, t$ it may be necessary to invoke the modification rules of table 4, as well as those associated with the appropriate maximal classical subgroup H, in order to interpret the results. However, corollary 2 guarantees that even after modification the tensor product decompositions are all multiplicity free.

In contrast to this the defining irrep (ω) of F_4 is not weight multiplicity free so that we cannot conclude that all tensor products involving (ω) will be multiplicity free. In fact we find

Table 7. The tensor product $(\omega) \times (\nu)$ for E_7.

$(1^2) \times (m, n, p, q, r, s, t)$	$= (m+1, n+1, p+1, q+1, r+1, s+1, t)$
	$+ (m+1, n+1, p+1, q+1, r+1, s, t+1)$
	$+ (m+1, n+1, p+1, q+1, r, s+1, t+1)$
	$+ (m+1, n+1, p+1, q, r+1, s+1, t+1)$
	$+ (m+1, n+1, p, q+1, r+1, s+1, t+1) + (m+1, n+1, p, q, r, s, t)$
	$+ (m+1, n, p+1, q+1, r+1, s+1, t+1) + (m+1, n, p+1, q, r, s, t)$
	$+ (m+1, n, p, q+1, r, s, t) + (m+1, n, p, q, r+1, s, t)$
	$+ (m+1, n, p, q, r, s+1, t)$
	$+ (m+1, n, p, q, r, s, t+1) + (m, n+1, p+1, q+1, r+1, s+1, t+1)$
	$+ (m, n+1, p+1, q, r, s, t) + (m, n+1, p, q+1, r, s, t)$
	$+ (m, n+1, p, q, r+1, s, t)$
	$+ (m, n+1, p, q, r, s+1, t) + (m, n+1, p, q, r, s, t+1)$
	$+ (m, n, p+1, q+1, r, s, t)$
	$+ (m, n, p+1, q, r+1, s, t) + (m, n, p+1, q, r, s+1, t)$
	$+ (m, n, p+1, q, r, s, t+1)$
	$+ (m, n, p, q+1, r+1, s, t) + (m, n, p, q+1, r, s+1, t)$
	$+ (m, n, p, q+1, r, s, t+1)$
	$+ (m, n, p, q, r+1, s+1, t) + (m, n, p, q, r+1, s, t+1)$
	$+ (m, n, p, q, r, s+1, t+1)$
	$+ (m, n, p, q, r, s-1, t-1) + (m, n, p, q, r-1, s, t-1)$
	$+ (m, n, p, q, r-1, s-1, t)$
	$+ (m, n, p, q-1, r, s, t-1) + (m, n, p, q-1, r, s-1, t)$
	$+ (m, n, p, q-1, r-1, s, t)$
	$+ (m, n, p-1, q, r, s, t-1) + (m, n, p-1, q, r, s-1, t)$
	$+ (m, n, p-1, q, r-1, s, t)$
	$+ (m, n, p-1, q-1, r, s, t) + (m, n-1, p, q, r, s, t-1)$
	$+ (m, n-1, p, q, r, s-1, t)$
	$+ (m, n-1, p, q, r-1, s, t) + (m, n-1, p, q-1, r, s, t)$
	$+ (m, n-1, p-1, q, r, s, t)$
	$+ (m, n-1, p-1, q-1, r-1, s-1, t-1) + (m-1, n, p, q, r, s, t-1)$
	$+ (m-1, n, p, q, r, s-1, t) + (m-1, n, p, q, r-1, s, t)$
	$+ (m-1, n, p, q-1, r, s, t)$
	$+ (m-1, n, p-1, q, r, s, t) + (m-1, n, p-1, q-1, r-1, s-1, t-1)$
	$+ (m-1, n-1, p, q, r, s, t) + (m-1, n-1, p, q-1, r-1, s-1, t-1)$
	$+ (m-1, n-1, p-1, q, r-1, s-1, t-1)$
	$+ (m-1, n-1, p-1, q-1, r, s-1, t-1)$
	$+ (m-1, n-1, p-1, q-1, r-1, s, t-1)$
	$+ (m-1, n-1, p-1, q-1, r-1, s-1, t)$

that

$$F_4 : c(\lambda; \omega, \nu) \in \{0, 1\} \qquad \text{for all } (\nu) \text{ with either } \begin{cases} \nu_4 = 0, \\ \nu_1 = \nu_2 + \nu_3 + \nu_4, \\ \text{or both.} \end{cases} \tag{4.2}$$

However, for other irreps (ν) tensor product mutiplicities of 2 will arise as is clear from the general results given in table 8.

The fact that the tensor products identified in (4.2) are multiplicity free can then be seen by noting that the cases $\nu_4 = 0$ and $\nu_1 = \nu_2 + \nu_3 + \nu_4$ correspond in the above tabulation to $(\nu) = (m, n, p, 0)$ and either $(\nu) = (n + p + r, n, p, r)$ or $(\nu) = (\Delta; n + p + r, n, p, r)$, respectively. The modifications leading to the required reduction of the maximum multiplicity from 2 to 1 in the two formulae of table 8 are then given by

Table 8. The tensor product $(\omega) \times (\nu)$ for F_4.

$$\begin{aligned}
(1) \times (m,n,p,r) = {} & (m+1,n,p,r) + (\Delta;m,n,p,r) + (\Delta;m,n,p,r-1) \\
& + (\Delta;m,n,p-1,r) + (\Delta;m,n,p-1,r-1) + (\Delta;m,n-1,p,r) \\
& + (\Delta;m,n-1,p,r-1) + (\Delta;m,n-1,p-1,r) + (\Delta;m,n-1,p-1,r-1) \\
& + (m,n+1,p,r) + (m,n,p+1,r) + (m,n,p,r+1) + 2(m,n,p,r) \\
& + (m,n,p,r-1) + (m,n,p-1,r) + (m,n-1,p,r) + (\Delta;m-1,n,p,r) \\
& + (\Delta;m-1,n,p,r-1) + (\Delta;m-1,n,p-1,r) + (\Delta;m-1,n,p-1,r-1) \\
& + (\Delta;m-1,n-1,p,r) + (\Delta;m-1,n-1,p,r-1) + (\Delta;m-1,n-1,p-1,r) \\
& + (\Delta;m-1,n-1,p-1,r-1) + (m-1,n,p,r) \\
(1) \times (\Delta;m,n,p,r) = {} & (\Delta;m+1,n,p,r) + (m+1,n+1,p+1,r+1) + (m+1,n+1,p+1,r) \\
& + (m+1,n+1,p,r+1) + (m+1,n+1,p,r) + (m+1,n,p+1,r+1) \\
& + (m+1,n,p+1,r) + (m+1,n,p,r+1) + (m+1,n,p,r) \\
& + (\Delta;m,n+1,p,r) + (\Delta;m,n,p+1,r) + (\Delta;m,n,p,r+1) \\
& + 2(\Delta;m,n,p,r) + (\Delta;m,n,p,r-1) + (\Delta;m,n,p-1,r) \\
& + (\Delta;m,n-1,p,r) + (m,n+1,p+1,r+1) + (m,n+1,p+1,r) \\
& + (m,n+1,p,r+1) + (m,n+1,p,r) + (m,n,p+1,r+1) \\
& + (m,n,p+1,r) + (m,n,p,r+1) + (m,n,p,r) \\
& + (\Delta;m-1,n,p,r)
\end{aligned}$$

$$\begin{aligned}
& r=0 : (m,n,p,r-1) = (m,n,p,-1) = -(m,n,p,0) = -(m,n,p,r), \\
& m=n+p+r : (\Delta;m,n-1,p-1,r-1) = (\Delta;n+p+r,n-1,p-1,r-1) \\
& \qquad = -(n+p+r,n,p,r) = -(m,n,p,r), \\
& m=n+p+r+1 : (m+1,n,p,r) = (n+p+r+2,n,p,r) \\
& \qquad = -(\Delta;n+p+r+1,n,p,r) = -(\Delta;m,n,p,r).
\end{aligned} \tag{4.3}$$

In the case of E_8 the defining irrep (ω) coincides with the adjoint irrep (θ). Quite generally, for any simple Lie algebra $\boldsymbol{g}$ and the corresponding Lie group G, in the case of the adjoint irrep (θ) it follows from proposition 4.1 of King and Wybourne [17] that

$$c(\lambda;\theta,\nu) \in \{0,1\} \qquad \text{for all } \lambda \neq \nu \tag{4.4a}$$

$$c(\nu;\theta,\nu) = b(\nu) \tag{4.4b}$$

where $b(\nu)$ is the breadth of ν, defined in (3.2) as the number of non-vanishing components of the Dynkin label $((a))$ of (ν).

At this juncture it is useful to introduce the idea of a *scaled fundamental irrep* which is an irrep (ν) with breadth $b(\nu)=1$ and Dynkin weight $w(\nu)=n$, with n any positive integer. It follows from the definition of breadth and Dynkin weight given in (3.2) that $\nu = n\omega_i$ for some $i \in \{1,2,\ldots,k\}$ and some positive integer n. In the Dynkin label notation $((a))$ such an irrep has a single non-vanishing component, namely the ith component $a_i = \nu(h_i) = n$, and in the standard, partition label notation (ν) the parts of (ν) are just the corresponding parts of the standard, partition label of the fundamental irrep (ω_i) all scaled by n.

Thus (4.4a), (4.4b) imply that the tensor products $(\theta)\times(\nu)$ will be multiplicity free if and only if (ν) is either a fundamental irrep (ω_i) or a scaled fundamental irrep $(n\omega_i)$ with $n > 1$. Such products may be evaluated explicitly for each exceptional Lie group for any particular value of n or indeed a range of different values. However, in carrying out this analysis we are aided by the following stability result which follows from corollary 4.

Proposition 1. *For any $\mu, \nu \in P^+$ with $\nu(h_i) \geqslant n_s(\mu)$ for some sufficiently large but positive finite integer $n_s(\mu)$, if*

$$(\mu) \times (\nu) = \sum_{\lambda \in P^+} c(\lambda;\mu,\nu)\,(\lambda), \tag{4.5}$$

Table 9. Explicit tensor products $(\theta) \times (n\omega_i)$ for G_2.

$(21) \times (n)$	$=$	$(n+2,1)_1 + (n+1,2)_3 + (n+1,1)_2 + (n+1)_1 + (n,1)_2 + (n)_1$
		$+ (n-1,1)_3 + (n-1)_2$
$(21) \times (2n,n)$	$=$	$(2n+2,n+1)_1 + (2n+1,n-1)_1 + (2n,n)_1 + (2n,n-1)_2$
		$+ (2n-1,n-2)_2 + (2n-2,n-1)_1$

then

$$(\mu) \times (\nu + \omega_i) = \sum_{\lambda \in P^+} c(\lambda;\mu,\nu)\,(\lambda + \omega_i). \tag{4.6}$$

Proof. It suffices to take $n_s(\mu)$ to be the maximum of those n_s defined for various λ but fixed μ and ν in corollary 4. □

In principle, it is not too difficult to evaluate $n_s(\mu)$ for any irrep (μ). This can be seen by noting that the Chevalley generators e_i, f_i, h_i generate an $sl(2) = A_1$ subalgebra of the Lie algebra $\boldsymbol{g}$ under consideration. On restriction from $\boldsymbol{g}$ to $sl(2)$ any irrep (μ) of $\boldsymbol{g}$ may be decomposed into a direct sum of irreps (m) of $sl(2)$ of dimension $m+1$. However every vector v in the corresponding $sl(2)$ module is annihilated by e_i^{m+1}. It follows that if we have the branching

$$\boldsymbol{g} \to sl(2) : (\mu) \to \sum_{m=m_{\min}}^{m_{\max}} b_m^{\mu}\,(m) \tag{4.7}$$

then

$$n_s(\mu) = m_{\max} + 1. \tag{4.8}$$

In the context of the particular tensor products $(\theta) \times (n\omega_i)$ under consideration here, proposition 1 implies that their decomposition becomes structurally stable with respect to the parameter n provided that the values of n are sufficiently large. In this sense there exists in each case a generic, n-dependent, decomposition of $(\theta) \times (n\omega_i)$ for each $i = 1, 2, \ldots, k$ for each of the exceptional groups. These generic decompositions are given in tables 9–13.

For lower values of n these decompositions remain valid, but are subject to modification rules. These rules leave the tensor products multiplicity free, as required by (4.4), but certain terms in the generic product are to be eliminated if n is too small. In the tabulation a term of the form $(\lambda)_r$ is to be retained if and only if $n \geqslant r$. The value, $n_s(\theta)$, of n at which the products stabilize to their generic form is thus the maximum value of the subscripts on the terms $(\lambda)_r$ appearing in any decomposition. It can be seen for example that the first G_2 product $(21) \times (n)$ stabilizes at $n = 3$, while all the other products stabilize at $n = 1$ or 2.

The decompositions of these tensor products were all determined through the use of SCHUR with n sufficiently large for stability to be observed. Then, by considering a range of smaller values of n, the stabilization value $n_s(\theta)$ was confirmed, as well as the values of r for each term $(\lambda)_r$. Finally, the generic result was checked dimensionally using Maple[5], with the dimension formula depending polynomially on the parameter n.

The same procedure was applied in the case of the results given in tables 5–8. The formulae were arrived at through the use either of SCHUR or the known weights of (ω), and they were then checked for accuracy using Maple to evaluate the dimension of each of the products as a multinomial in the relevant parameters $m, n, p, \ldots, t$.

[5] Maple, Waterloo Maple Software, Waterloo, Ontario, Canada.

Table 10. Explicit tensor products $(\theta) \times (n\omega_i)$, with $i = 1, 2, 4$, $(\theta) \times (2n\omega_3)$, $(\theta) \times ((2n+1)\omega_3)$ for F_4.

$(1^2) \times (n)$	$=$	$(n+1, 1)_1 + (\Delta; n)_1 + (n, 1^2)_2 + (n)_1 + (\Delta; n-1)_2 + (n-1, 1)_2$
$(1^2) \times (n^2)$	$=$	$(n+1, n+1)_1 + (n+1, n, 1)_1 + (n+1, n-1)_1 + (n^2)_1$
		$+ (n, n-1, 1)_2 + (n-1, n-1)_1$
$(1^2) \times (2n, n^2)$	$=$	$(2n+1, n+1, n)_1 + (2n+1, n^2, 1)_1 + (2n+1, n, n-1)_1$
		$+ (\Delta; 2n, n-1, n-1)_1 + (2n, n+1, n-1)_1 + (2n, n^2)_1$
		$+ (2n, n, n-1, 1)_2 + (2n, n-1, n-1)_1 + (2n-1, n, n-1)_1$
$(1^2) \times (3n, n^3)$	$=$	$(3n+1, n+1, n^2)_1 + (3n+1, n^3)_1 + (3n+1, n^2, n-1)_1$
		$+ (\Delta; 3n, n, n-1, n-1)_1 + (\Delta; 3n, (n-1)^3)_1 + (3n, n+1, n, n-1)_1$
		$+ (3n, n^3)_1 + (3n, n^2, n-1)_1 + (3n, n, n-1, n-1)_1$
		$+ (\Delta; 3n-1, (n-1)^3)_1 + (3n-1, n^2, n-1)_1$
$(1^2) \times (\Delta; 3n+1, n^3)$	$=$	$(\Delta; 3n+2, n+1, n^2)_1 + (\Delta; 3n+2, n^3)_1 + (\Delta; 3n+2, n^2, n-1)_1$
		$+ (3n+2, n+1, n^2)_1 + (3n+2, n^3)_1 + (\Delta; 3n+1, n+1, n, n-1)_1$
		$+ (\Delta; 3n+1, n^3)_1 + (\Delta; 3n+1, n^2, n-1)_1 + (\Delta; 3n+1, n, n-1, n-1)_1$
		$+ (3n+1, n^3)_1 + (\Delta; 3n, n^2, n-1)_1$

Table 11. Explicit tensor products $(\theta) \times (n\omega_i)$ for E_6.

$(2:0) \times (n:n)$	$=$	$(n+2:n)_1 + (n+1:n, 1^3)_1 + (n:n)_1 + (n:n-1, 1)_2$
$(2:0) \times (n:n^5)$	$=$	$(n+2:n^5)_1 + (n+1:n^2, (n-1)^3)_1 + (n:n^5)_1 + (n:(n-1)^4, n-2)_2$
$(2:0) \times (2n:0)$	$=$	$(2n+2:0)_1 + (2n+1:1^3)_1 + (2n:21^4)_1 + (2n:0)_1$
		$+ (2n-1:1^3)_2 + (2n-2:0)_1$
$(2:0) \times (2n:n^2)$	$=$	$(2n+2:n^2)_1 + (2n+1:n+1, n, 1^2)_1 + (2n+1:n^2, 1^3)_1$
		$+ (2n:n+1, n-1)_1 + (2n:n^2)_1 + (2n:n, n-1, 1)_2 + (2n-1:n^2, 1^3)_1$
$(2:0) \times (2n:n^4)$	$=$	$(2n+2:n^4)_1 + (2n+1:(n+1)^2, n^2, 1)_1 + (2n+1:n, (n-1)^3)_1$
		$+ (2n:(n+1)^4, 2)_1 + (2n:n^4)_1 + (2n:n^3, n-1, 1)_2 + (2n-1:n, (n-1)^3)_1$
$(2:0) \times (3n:n^3)$	$=$	$(3n+2:n^3)_1 + (3n+1:(n+1)^2, n, 1)_1 + (3n+1:n+1, n^2, 1^2)_1$
		$+ (3n+1:(n-1)^3)_1 + (3n:(n+1)^3, 2, 1)_1 + (3n:n+1, n, n-1)_1$
		$+ (3n:n^3)_1 + (3n:n^2, n-1, 1)_2 + (3n-1:n+1, n^2, 1^2)_1$
		$+ (3n-1:(n-1)^3)_1$

5. Tensor products of pairs of fundamental irreps

The results of the previous section give the complete decomposition of all multiplicity-free tensor products of the defining and the adjoint irreps with any other irrep for each exceptional Lie group. Both the defining (as well as its contragredient in the case of E_6) and the adjoint irrep are fundamental irreps in the case of the exceptional Lie groups. But there remain other fundamental irreps and, of course, scaled fundamental irreps to be dealt with, let alone irreps of greater breadth. To exhaust all possibilities for multiplicity-free tensor products we proceed by considering a succession of types of tensor product $(\mu) \times (\nu)$ characterized by the values of the breadth and Dynkin weight of (μ) and (ν). The various cases are defined in table 14.

The significance of this list of cases is that passing from one case to another in search of multiplicity-free tensor products allows one to invoke the crucial corollary 3 that was pointed out by Stembridge [19]. This corollary has the following rather obvious consequence.

Corollary 5. *Let $\mu, \nu, \pi \in P^+$. If the tensor product $(\mu) \times (\nu)$ is not multiplicity free, then (i) neither is the tensor product $(\mu) \times (\nu + \omega_i)$ for any $i = 1, 2, \ldots, k$, (ii) nor any tensor product of the form $(\mu) \times (\nu + \pi)$.*

Proof. By hypothesis there exists $\lambda \in P^+$ such that $c(\lambda; \mu, \nu) > 1$. Corollary 3 then implies that $c(\lambda + \omega_i; \mu, \nu + \omega_i) \geqslant c(\lambda; \mu, \nu) > 1$, proving part (i). Since $\pi = \sum_{i=1}^{k} a_i\omega_i$ with a_i a

Table 12. Explicit tensor products $(\theta) \times (n\omega_i)$ for E_7.

$(21^6) \times (2n)$	$=$	$(2n+2, 1^6)_1 + (2n+1, 21^5)_1 + (2n+1, 1^3)_1 + (2n, 1^4)_2$
		$+ (2n)_1 + (2n-1, 1)_1$
$(21^6) \times (n^2)$	$=$	$(n+2, n+1, 1^5)_1 + (n+1, n-1)_1 + (n^2)_1 + (n, n-1, 1)_2$
$(21^6) \times (2n, n^2)$	$=$	$(2n+2, (n+1)^2, 1^4)_1 + (2n+1, n^2, 1^3)_1 + (2n+1, n, n-1)_1$
		$+ (2n, n+1, n-1)_1 + (2n, n^2, 1^4)_1 + (2n, n^2)_1$
		$+ (2n, n, n-1, 1)_2$
$(21^6) \times (2n, n^6)$	$=$	$(2n+2, (n+1)^6)_1 + (2n+1, n^5, n-1)_1 + (2n, n^6)_1 + (2n, n^2, (n-1)^4)_1$
		$+ (2n-1, (n-1)^5, n-2)_2 + (2n-2, (n-1)^6)_1$
$(21^6) \times (3n, n^3)$	$=$	$(3n+2, (n+1)^3, 1^3)_1 + (3n+1, n+1, n^2, 1^2)_1 + (3n+1, n^3, 1^3)_1$
		$+ (3n+1, n^2, n-1)_1 + (3n, n+1, n^2, 1^3)_1 + (3n, n+1, n, n-1)_1$
		$+ (3n, n^3)_1 + (3n, n^2, n-1, 1)_2 + (3n-1, (n-1)^3)_1$
$(21^6) \times (3n, n^5)$	$=$	$(3n+2, (n+1)^5, 1)_1 + (3n+1, (n+1)^5, 2)_1 + (3n+1, (n+1)^2, n^3, 1)_1$
		$+ (3n+1, n^4, n-1)_1 + (3n, n^5)_1 + (3n, n^4, n-1, 1)_2$
		$+ (3n, n, (n-1)^4)_1 + (3n-1, n^5, 1)_1 + (3n-1, n^2, (n-1)^3)_1$
$(21^6) \times (4n, n^4)$	$=$	$(4n+2, (n+1)^4, 1^2)_1 + (4n+1, (n+1)^4, 21)_1 + (4n+1, (n+1)^2, n^2, 1)_1$
		$+ (4n+1, n+1, n^3, 1^2)_1 + (4n+1, n^3, n-1)_1 + (4n, (n+1)^2, n^2, 1^2)_1$
		$+ (4n, n+1, n^2, n-1)_1 + (4n, n^4)_1 + (4n, n^3, n-1, 1)_2$
		$+ (4n, (n-1)^4)_1 + (4n-1, n^4, 1)_1 + (4n-1, n, (n-1)^3)_1$

Table 13. Explicit tensor products $(\theta) \times (n\omega_i)$ for E_8.

$(21^7) \times (2n, n)$	$=$	$(2n+2, n+1, 1^6)_1 + (2n+1, n, 1^5)_1 + (2n+1, n-1)_1$
		$+ (2n, n, 1^6)_1 + (2n, n)_1 + (2n, n-1, 1)_2$
$(21^7) \times (2n, n^7)$	$=$	$(2n+2, (n+1)^7)_1 + (2n+1, n^6, n-1)_1 + (2n, n^7)_1$
		$+ (2n, n, (n-1)^6)_1 + (2n-1, (n-1)^6, n-2)_2 + (2n-2, (n-1)^7)_1$
$(21^7) \times (3n)$	$=$	$(3n+2, 1^7)_1 + (3n+1, 21^6)_1 + (3n+1, 1^5)_1 + (3n+1, 1^2)_1$
		$+ (3n, 1^6)_1 + (3n, 1^3)_2 + (3n)_1 + (3n-1, 1)_1$
$(21^7) \times (3n, n^6)$	$=$	$(3n+2, (n+1)^6, 1)_1 + (3n+1, (n+1)^6, 2)_1 + (3n+1, n+1, n^5, 1)_1$
		$+ (3n+1, n^5, n-1)_1 + (3n, n^6)_1 + (3n, n^5, n-1, 1)_2$
		$+ (3n, (n-1)^6)_1 + (3n-1, n^6, 1)_1 + (3n-1, n, (n-1)^5)_1$
$(21^7) \times (4n, n^2)$	$=$	$(4n+2, (n+1)^2, 1^5)_1 + (4n+1, n+1, n, 1^4)_1 + (4n+1, n^2, 1^5)_1$
		$+ (4n+1, n^2, 1^2)_1 + (4n+1, n, n-1)_1 + (4n, n+1, n, 1^5)_1$
		$+ (4n, n+1, n-1)_1 + (4n, n^2, 1^3)_1 + (4n, n^2)_1$
		$+ (4n, n, n-1, 1)_2 + (4n-1, (n-1)^2)_1$
$(21^7) \times (4n, n^5)$	$=$	$(4n+2, (n+1)^5, 1^2)_1 + (4n+1, (n+1)^5, 21)_1 + (4n+1, n+1, n^4, 1)_1$
		$+ (4n+1, n^5, 1^2)_1 + (4n+1, n^4, n-1)_1 + (4n, n+1, n^4, 1^2)_1$
		$+ (4n, n^5)_1 + (4n, n^4, n-1, 1)_2 + (4n, n^2, (n-1)^3)_1$
		$+ (4n-1, n^5, 1)_1 + (4n-1, (n-1)^5)_1$
$(21^7) \times (5n, n^4)$	$=$	$(5n+2, (n+1)^4, 1^3)_1 + (5n+1, (n+1)^4, 21^2)_1 + (5n+1, (n+1)^2, n^2, 1^3)_1$
		$+ (5n+1, n+1, n^3, 1)_1 + (5n+1, n^4, 1^2)_1 + (5n+1, n^3, n-1)_1$
		$+ (5n, n+1, n^3, 1^2)_1 + (5n, n^4, 1^3)_1 + (5n, n^4)_1$
		$+ (5n, n^3, n-1, 1)_2 + (5n, n, (n-1)^3)_1 + (5n-1, n^4, 1)_1$
		$+ (5n-1, n^2, (n-1)^2)_1$
$(21^7) \times (6n, n^3)$	$=$	$(6n+2, (n+1)^3, 1^4)_1 + (6n+1, (n+1)^3, 21^3)_1 + (6n+1, (n+1)^2, n, 1^3)_1$
		$+ (6n+1, n+1, n^2, 1^4)_1 + (6n+1, n+1, n^2, 1)_1 + (6n+1, n^3, 1^2)_1$
		$+ (6n+1, n^2, n-1)_1 + (6n, (n+1)^2, n, 1^4)_1 + (6n, n+1, n^2, 1^2)_1$
		$+ (6n, n+1, n, n-1)_1 + (6n, n^3, 1^3)_1 + (6n, n^3)_1$
		$+ (6n, n^2, n-1, 1)_2 + (6n, (n-1)^3)_1 + (6n-1, n^3, 1)_1$
		$+ (6n-1, n, (n-1)^2)_1$

Table 14. Classification of types of tensor product $(\mu) \times (\nu)$ characterized by the breadth and Dynkin weight of (μ) and (ν).

Case 1	$b(\mu) = 1$	$w(\mu) = 1$	$b(\nu) = 1$	$w(\nu) = 1$
Case 2	$b(\mu) = 1$	$w(\mu) = 1$	$b(\nu) = 1$	$w(\nu) = n > 1$
Case 3	$b(\mu) = 1$	$w(\mu) = m > 1$	$b(\nu) = 1$	$w(\nu) = n > 1$
Case 4	$b(\mu) = 1$	$w(\mu) = 1$	$b(\nu) > 1$	$w(\nu) = n > 1$
Case 5	$b(\mu) = 1$	$w(\mu) = m > 1$	$b(\nu) > 1$	$w(\nu) = n > 1$
Case 6	$b(\mu) > 1$	$w(\mu) = m > 1$	$b(\nu) > 1$	$w(\nu) = n > 1$

Table 15. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, \omega_j)$ for G_2.

		(21)	(1)
10	(21)	1	1
01	(1)	1	1

Table 16. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, \omega_j)$ for F_4.

		(1^2)	(21^2)	$(\Delta; 1)$	(1)
1000	(1^2)	1	1	1	1
0100	(21^2)	1	3	2	1
0010	$(\Delta; 1)$	1	2	2	1
0001	(1)	1	1	1	1

non-negative integer for $i = 1, 2, \ldots, k$, part (i) may then be used precisely $w(\pi) = \sum_{i=1}^{k} a_i$ times, by adding to μ a succession of a_i copies of ω_i for $i = 1, 2, \ldots, k$, to conclude that $c(\lambda + \pi; \mu, \nu + \pi) \geqslant c(\lambda; \mu, \nu) > 1$, thereby proving part (ii). □

In the present context the significance of this corollary is that passing from (ν) to $(\nu + \omega_i)$ is equivalent to adding 1 to the ith component of the Dynkin label $((a))$ of (ν). This will certainly increase the Dynkin weight by 1, and will also increase the breadth by 1 if the ith component of $((a))$ was originally zero. The same is true in passing from (μ) to $(\mu + \omega_i)$ for all $i = 1, 2, \ldots, k$. These increases of breadth and Dynkin weight are precisely what are involved in moving from one case to another in the tabulated sequence of cases 1–6, and in moving from m to $m + 1$ or n to $n + 1$ within the appropriate cases.

Here case 1 corresponds to the most important case of all, namely the tensor products of all possible pairs of fundamental irreps $(\omega_i) \times (\omega_j)$ with $i, j \in \{1, 2, \ldots, k\}$. These tensor products may be readily determined and we display the resulting data in the form of a matrix of positive integers. Each entry is the *maximum multiplicity* appearing in the decomposition of the tensor product of the irreps $(\omega_i) \times (\omega_j)$. Row irreps (ω_i) are labelled using both Dynkin and standard partition labels, while column irreps (ω_j) are just labelled using standard partition labels. These results are given in tables 15–19. They establish those case 1 tensor products $(\mu) \times (\nu)$ with $b(\mu) = w(\mu) = b(\nu) = w(\nu) = 1$ that are multiplicity free, namely those associated with an entry 1 in tables 15–19. All other entries are associated with non-multiplicity-free products.

Almost all of the multiplicity-free tensor products that are identified through the entry 1 in these tables have already been encountered in our analysis of products involving the defining irrep, its contragredient and the adjoint irrep. There remain just two new ones:

$$E_7: (2) \times (2) = (42^6) + (42^2 1^4) + (41^4) + (4) + (31^5) + (31) + (2^2) + (21^6) + (21^2) + (0); \quad (5.1a)$$

$$E_8: (21) \times (21) = (42^7) + (421^6) + (42) + (41^5) + (41^2) + (31^6) + (3) + (21^7) + (21) + (0). \quad (5.1b)$$

Table 17. Maximum values of the tensor product multiplicities $c(\lambda;\omega_i,\omega_j)$ for E_6.

		(1:1)	(2:1^2)	(3:1^3)	(2:1^4)	(1:1^5)	(2:0)
100 000	(1:1)	1	1	1	1	1	1
010 000	(2:1^2)	1	2	2	2	1	1
001 000	(3:1^3)	1	2	4	2	1	1
000 100	(2:1^4)	1	2	2	2	1	1
000 010	(1:1^5)	1	1	1	1	1	1
000 001	(2:0)	1	1	1	1	1	1

Table 18. Maximum values of the tensor product multiplicities $c(\lambda;\omega_i,\omega_j)$ for E_7.

		(21^6)	(31^5)	(41^4)	(31^3)	(21^2)	(1^2)	(2)
1000 000	(21^6)	1	1	1	1	1	1	1
0100 000	(31^5)	1	3	5	3	2	1	2
0010 000	(41^4)	1	5	12	5	3	1	2
0001 000	(31^3)	1	3	5	4	2	1	2
0000 100	(21^2)	1	2	3	2	2	1	2
0000 010	(1^2)	1	1	1	1	1	1	1
0000 001	(2)	1	2	2	2	2	1	1

Table 19. Maximum values of the tensor product multiplicities $c(\lambda;\omega_i,\omega_j)$ for E_8.

		(21^7)	(31^6)	(41^5)	(51^4)	(61^3)	(41^2)	(21)	(3)
10 000 000	(21^7)	1	1	1	1	1	1	1	1
01 000 000	(31^6)	1	3	4	5	7	4	2	3
00 100 000	(41^5)	1	4	9	15	27	10	2	5
00 010 000	(51^4)	1	5	15	40	81	18	2	6
00 001 000	(61^3)	1	7	27	81	214	33	3	10
00 000 100	(41^2)	1	4	10	18	33	10	2	5
00 000 010	(21)	1	2	2	2	3	2	1	2
00 000 001	(3)	1	3	5	6	10	5	2	3

6. Tensor products of fundamental irreps with scaled fundamental irreps

In case 2 we consider the cases where all the column labels involved in case 1 are scaled by n, so that the corresponding non-vanishing component of the Dynkin label for (ν) is now n, with $n \geqslant 2$, that is tensor products of the form $(\mu)\times(\nu) = (\omega_i)\times(n\omega_j)$ for which $b(\mu) = b(\nu) = w(\mu) = 1$ and $w(\nu) = n$ with $n \geqslant 2$. Products of this type with $(\mu) = (\omega)$, $(\overline{\omega})$ and (θ), corresponding to the defining, its contragredient and the adjoint irreps, have already been considered. The others are new. Our results are again displayed as matrices in tables 20–24.

In these matrices an entry $*$ indicates that the product is not multiplicity free even for $n = 1$, as evidenced by the case 1 data of tables 15–19. Corollary 5 then implies that it is not multiplicity free for any $n \geqslant 1$. Similarly, an entry $**$ indicates that explicit calculation of the product has shown it to be not multiplicity free for $n = 2$. As before, corollary 5 then implies that it is not multiplicity free for any $n \geqslant 2$. In contrast to this, an entry 1 indicates that the decomposition of the tensor product is found to be multiplicity free not just for $n = 2$ but for all $n \geqslant 2$. The latter is a consequence of the stabilization of such products with respect to $\nu(h_j) = n\omega_j(h_j) = n$ and proposition 1, together with the observation that in every case the stability onset parameter is found to be given by $n_s(\omega_i) = 2$.

Table 20. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, n\omega_j)$ for G_2.

		$(2n, n)$	(n)
10	(2, 1)	1	1
01	(1)	1	1

Table 21. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, n\omega_j)$ for F_4.

		(n^2)	$(2n, n^2)$	$(\frac{3n}{2}, \frac{n}{2}^3)$	(n)
1000	(1^2)	1	1	1	1
0100	(21^2)	**	*	*	**
0010	$(\frac{3}{2}\frac{1}{2}^3)$	1	*	*	**
0001	(1)	1	1	1	1

Table 22. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, n\omega_j)$ for E_6.

		$(n : n)$	$(2n : n^2)$	$(3n : n^3)$	$(2n : n^4)$	$(n : n^5)$	$(2n : 0)$
100 000	(1 : 1)	1	1	1	1	1	1
010 000	$(2 : 1^2)$	1	*	*	*	1	1
001 000	$(3 : 1^3)$	**	*	*	*	**	**
000 100	$(2 : 1^4)$	1	*	*	*	1	1
000 010	$(1 : 1^5)$	1	1	1	1	1	1
000 001	(2 : 0)	1	1	1	1	1	1

Table 23. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, n\omega_j)$ for E_7.

		$(2n, n^6)$	$(3n, n^5)$	$(4n, n^4)$	$(3n, n^3)$	$(2n, n^2)$	(n^2)	$(2n)$
1000 000	(21^6)	1	1	1	1	1	1	1
0100 000	(31^5)	**	*	*	*	*	**	*
0010 000	(41^4)	**	*	*	*	*	**	*
0001 000	(31^3)	**	*	*	*	*	**	*
0000 100	(21^2)	1	**	**	**	**	1	*
0000 010	(1^2)	1	1	1	1	1	1	1
0000 001	(2)	1	*	*	*	*	1	1

Table 24. Maximum values of the tensor product multiplicities $c(\lambda; \omega_i, n\omega_j)$ for E_8.

		$(2n, n^7)$	$(3n, n^6)$	$(4n, n^5)$	$(5n, n^4)$	$(6n, n^3)$	$(4n, n^2)$	$(2n, n)$	$(3n)$
10 000 000	(21^7)	1	1	1	1	1	1	1	1
01 000 000	(31^6)	**	*	*	*	*	*	*	*
00 100 000	(41^5)	**	*	*	*	*	*	*	*
00 010 000	(51^4)	**	*	*	*	*	*	*	*
00 001 000	(61^3)	**	*	*	*	*	*	*	*
00 000 100	(41^2)	**	*	*	*	*	*	*	*
00 000 010	(21)	1	*	*	*	*	*	1	*
00 000 001	(3)	**	*	*	*	*	*	*	*

In the case of those tensor products that are indicated to be multiplicity free by means of an entry 1 the conclusions have been tested by explicit calculations for values of n ranging from $n = 1$ up to values that are sufficiently large for the product to be generic, in the sense that the

Table 25. Further explicit multiplicity-free tensor products $(\omega_i) \times (n\omega_j)$ for F_4.

$(\Delta; 1) \times (n^2)$	$=$	$(\Delta; n+1, n)_1 + (\Delta; n+1, n-1)_1 + (n+1, n)_1 + (n+1, n-1, 1)_2$
		$+ (n+1, n-1)_1 + (\Delta; n, n-1)_1 + (\Delta; n, n-2)_2 + (n, n-1)_1$
		$+ (\Delta; n-1, n-2)_2$

Table 26. Further explicit multiplicity-free tensor products $(\omega_i) \times (n\omega_j)$ for E_6.

$(2:1^2) \times (n:n)$	$=$	$(n+2:n+1, 1)_1 + (n+2:n, 1^2)_1 + (n+1:n+1, 1^4)_1$
		$+ (n+1:n, 21^3)_2 + (n+1:n-1)_1 + (n:n-1, 1^3)_2$
$(2:1^4) \times (n:n)$	$=$	$(n+2:n+1, 1^3)_1 + (n+2:n, 1^4)_1 + (n+1:n, 1)_1$
		$+ (n+1:n-1, 1^2)_2 + (n:n, 1^4)_1 + (n:n-2)_2$
$(2:1^2) \times (2n:0)$	$=$	$(2n+2:1^2)_1 + (2n+1:21^3)_1 + (2n+1:1^5)_1$
		$+ (2n:2^3 1^2)_2 + (2n:2)_1 + (2n:1^2)_1$
		$+ (2n-1:21^3)_2 + (2n-1:1^5)_1 + (2n-2:1^2)_2$
$(2:1^4) \times (2n:0)$	$=$	$(2n+2:1^4)_1 + (2n+1:2^2 1^3)_1 + (2n+1:1)_1$
		$+ (2n:2^5)_1 + (2n:21^2)_2 + (2n:1^4)_1$
		$+ (2n-1:2^2 1^3)_2 + (2n-1:1)_1 + (2n-2:1^4)_2$

Table 27. Further explicit multiplicity-free tensor products $(\omega_i) \times (n\omega_j)$ for E_7.

$(2) \times (2n)$	$=$	$(2n+2, 2^6)_1 + (2n+2, 2^2\ 1^4)_1 + (2n+2,\ 1^4)_1 + (2n+2)_1$
		$+ (2n+1, 2^3\ 1^3)_2 + (2n+1, 21^3)_2 + (2n+1, 1^5)_1 + (2n+1, 1)_1$
		$+ (2n, 21^4)_2 + (2n, 2)_1 + (2n, 1^6)_1 + (2n, 1^2)_1 + (2n-1, 1^3)_2$
		$+ (2n-2)_1$
$(2) \times (n^2)$	$=$	$(n+2, n, 1^4)_1 + (n+2, n)_1 + (n+1, n, 1^5)_1 + (n+1, n, 1)_1$
		$+ (n+1, n-1, 1^2)_2 + (n, n-2)_2$
$(2) \times (2n, n^6)$	$=$	$(2n+2, n^6)_1 + (2n+1, n+1, n^5)_1 + (2n+1, n^3, (n-1)^3)_1$
		$+ (2n, n, (n-1)^4, n-2)_2 + (2n, (n-1)^6)_1 + (2n-1, n, (n-1)^5)_1$
		$+ (2n-1, (n-1)^3, (n-2)^3)_2 + (2n-2, (n-2)^6)_2$
$(21^2) \times (2n, n^6)$	$=$	$(2n+2, (n+1)^2, n^4)_1 + (2n+1, n^5, n-1)_1 + (2n+1, n, (n-1)^5)_1$
		$+ (2n, n+1, (n-1)^5)_1 + (2n, n^6)_1 + (2n, n^2, (n-1)^4)_1$
		$+ (2n, (n-1)^4, (n-2)^2)_2 + (2n-1, (n-1)^5, n-2)_2 + (2n-1, n-1, (n-2)^5)_2$
		$+ (2n-2, (n-1)^2, (n-2)^4)_2$
$(21^2) \times (n^2)$	$=$	$(n+2, n+1, 1^5)_1 + (n+2, n+1, 1)_1 + (n+2, n, 21^4)_2 + (n+2, n, 1^2)_1$
		$+ (n+1, n-1, 1^4)_2 + (n+1, n-1)_1 + (n^2)_1 + (n, n-1, 1^5)_2 + (n, n-1, 1)_2$

decomposition of the product has a structurally stable form with terms having an identifiable dependence on n, with all terms multiplicity free for all n. The explicit expansions for all these case 2 multiplicity-free products are given in tables 25– 28, save for the two E_6 products $(2:1^2) \times (n:n^5)$ and $(2:1^4) \times (n:n^5)$ which may be obtained from $(2:1^4) \times (n:n)$ and $(2:1^2) \times (n:n)$, respectively, by taking their contragredients. The convention on the subscripts appearing in tables 25–28 is as before. That is, a subscript r on any term $(\lambda)_r$ indicates that the irrep (λ) is present in the decomposition of the tensor product for all $n \geqslant r$. Since the only values of r that occur are $r = 1$ and 2, and these values appear in each product, it follows that each product is stable with respect to n, in the sense of proposition 1, for all $n \geqslant 2$.

Once again, the decompositions of these tensor products were all determined through the use of SCHUR with n sufficiently large for stability to be observed. Then, by considering a range of smaller values of n, the stabilization value $n_s(\omega_i)$ was confirmed, as well as the values

Table 28. Further explicit multiplicity-free tensor products $(\omega_i) \times (n\omega_j)$ for E_8.

$(21) \times (2n, n)$	=	$(2n+2, n+1, 2^6)_1 + (2n+2, n+1, 1^6)_1 + (2n+2, n+1)_1$
		$+ (2n+2, n, 21^5)_2 + (2n+2, n, 1^4)_1 + (2n+2, n, 1)_1$
		$+ (2n+1, n, 1^5)_1 + (2n+1, n-1, 1^6)_2 + (2n+1, n-1, 1^3)_2$
		$+ (2n+1, n-1)_1 + (2n, n, 1^6)_1 + (2n, n)_1$
		$+ (2n, n-1, 1^4)_2 + (2n, n-1, 1)_2 + (2n-1, n-2)_2$
		$+ (2n-2, n-1)_1$
$(21) \times (2n, n^7)$	=	$(2n+2, n+1, n^6)_1 + (2n+1, n^6, n-1)_1 + (2n+1, (n-1)^7)_1$
		$+ (2n, n^7)_1 + (2n, n, (n-1)^6)_1 + (2n, (n-1)^5, (n-2)^2)_2$
		$+ (2n-1, (n-1)^6, n-2)_2 + (2n-1, (n-2)^7)_2$
		$+ (2n-2, n-1, (n-2)^6)_2$

Table 29. Multiplicity-free tensor products $(m\omega_i) \times (n\omega_j)$ with $m, n \geqslant 2$.

G_2:	$(2) \times (2n, n)$;
F_4:	$(2) \times (n^2)$;
E_6:	$(m : m) \times (n : n)$, $(m : m^5) \times (n : n^5)$, $(m : m) \times (n : n^5)$,
	$(m : m) \times (2n : n^2)$, $(m : m) \times (2n : n^4)$, $(m : m) \times (2n : 0)$,
	$(m : m^5) \times (2n : n^2)$, $(m : m^5) \times (2n : n^4)$, $(m : m^5) \times (2n : 0)$;
E_7:	$(2^2) \times (2n, n^2)$, $(m^2) \times (2n, n^6)$, $(m^2) \times (n^2)$, $(m^2) \times (2n)$.

of r for each term $(\lambda)_r$. Finally, the generic result was checked dimensionally using Maple, with the dimension formula depending polynomially on the parameter n.

7. Tensor products of scaled fundamental irreps with scaled fundamental irreps

In case 3 tensor products of scaled fundamental irreps with scaled fundamental irreps are considered, that is tensor products $(\mu) \times (\nu) = (m\omega_i) \times (n\omega_j)$ for which $b(\mu) = b(\nu) = 1$ and $w(\mu) = m$ and $w(\nu) = n$ with $m, n \geqslant 2$. Given the weakly monotonic increasing nature of the maximum tensor product multiplicity with respect to both m and n, in order to determine the set of all case 3 multiplicity-free tensor products it is only necessary to confine attention to those case 2 tensor products for which the $(\omega_i) \times (n\omega_j)$ and $(\omega_j) \times (n\omega_i)$ matrix elements in tables 20–24 are both 1. The complete list of case 3 tensor products that are found to be multiplicity free is given in table 29.

In the three cases for which m is limited to the value 2, the corresponding tensor products, valid for all $n \geqslant 2$, take the form

$$G_2: \quad (2) \times (2n, n) = (2n+2, n)_1 + (2n+1, n)_1 + (2n+1, n-1)_1 + (2n, n)_1 + (2n, n-1)_1 + (2n, n-2)_2 + (2n-1, n-1)_1 + (2n-1, n-2)_2 + (2n-2, n-2)_2. \tag{7.1}$$

$$F_4: \quad (2) \times (n^2) = (n+2, n)_1 + (\Delta; n+1, n-1)_1 + (n+1, n, 1)_1 + (n+1, n-1, 1^2)_2 + (n+1, n-1)_1 + (\Delta; n, n-1)_1 + (\Delta; n, n-2)_2 + (n^2)_1 + (n, n-1, 1)_2 + (n, n-2)_2. \tag{7.2}$$

Table 30. Upper limits of summation for the E_6 tensor product expansions (7.4a)–(7.4e).

$(m:m)\times(n:n)$	$A=\min(m,n)$ $B=\min(m-a,n-a)$
$(m:m)\times(n:n^5)$	$A=\min(m,n)$ $B=\min(m-a,n-a)$
$(m:m)\times(2n:0)$	$A=\min([m/2],n)$ $B=\min(m-2a,n-a)$ $C=\min(m-2a-b,n-a-b)$
$(m:m)\times(2n:n^2)$	$A=\min([m/2],n)$ $B=\min(m-2a,n-a)$ $C=\min([(m-2a-b)/2],n-a-b]$ $D=\min(m-2a-b-2c,n-a-b-c)$ $E=\min(m-2a-b-2c-d,n-a-b-c-d)$
$(m:m)\times(2n:n^4)$	$A=\min([m/2],n)$ $B=\min(m-2a,n-a)$ $C=\min([(m-2a-b)/2],n-a-b]$ $D=\min(m-2a-b-2c,n-a-b-c)$ $E=\min(m-2a-b-2c-d,n-a-b-c-d)$

$$\begin{aligned}E_7:\quad (2^2)\times(2n,n^2) &= (2n+2,n+2,n,2^4)_2+(2n+2,n+2,n,1^4)_1+(2n+2,n+2,n)_1\\ &+(2n+2,(n+1)^2,1^4)_1+(2n+2,n+1,n,21^3)_2+(2n+2,n+1,n,1)_2\\ &+(2n+2,n^2,2)_2+(2n+1,n^2,1^3)_1+(2n+1,n,n-1,1^4)_2\\ &+(2n+1,n,n-1)_1+(2n+1,(n-1)^2,1)_2+(2n,n+1,n-1,1^4)_2\\ &+(2n,n+1,n-1)_1+(2n,n^2,1^4)_1+(2n,n^2)_1+(2n,n,n-1,1)_2\\ &+(2n,(n-2)^2)_2+(2n-1,n-1,n-2)_2+(2n-2,n,n-2)_2. \end{aligned} \tag{7.3}$$

As can be seen from the values of the subscripts r, it is found once again that each of these decompositions is stable as a function of n for all $n \geqslant 2$. Once more the formulae, arrived at through the use of SCHUR, have been checked dimensionally using Maple.

In the remaining cases, the corresponding case 3 m- and n-dependent expansions of these tensor products are given for E_6 by

$$(m:m)\times(n:n)=\sum_{a,b=0}^{A,B}(m+n-a:m+n-a-b,a+b,a^3) \tag{7.4a}$$

$$(m:m)\times(n:n^5)=\sum_{a,b=0}^{A,B}(m+n-2a:m+n-2a-2b,(n-a-b)^4) \tag{7.4b}$$

$$(m:m)\times(2n:0)=\sum_{a,b,c=0}^{A,B,C}(m+2n-2a-2b-c:m-a,a+c,c^2,0) \tag{7.4c}$$

$$\begin{aligned}(m:m)\times(2n:n^2)=\sum_{a,b,c,d,e=0}^{A,B,C,D,E}&(m+2n-2a-b-c-d:m+n\\ &-2a-2b-c-e,n-b+c,a+c+d+e,a+c+d,c+d)\end{aligned} \tag{7.4d}$$

$$\begin{aligned}(m:m)\times(2n:n^4)=\sum_{a,b,c,d,e=0}^{A,B,C,D,E}&(m+2n-2a-2b-c-d:m+n\\ &-3a-b-2c-d-e,n-a,n-a-d,n-a-c-d,b+e)\end{aligned} \tag{7.4e}$$

where the upper limits A, B, C of the summation parameters are given in table 30.

Table 31. Upper limits of summation for the E_7 tensor product expansions (7.5*a*)–(7.5*c*).

$(m^2)\times(2n,n^6)$	$A=\min([m/2],n)$
	$B=\min(m-2a,n-a)$
	$C=\min(m-2a-b,n-a-b)$
$(m^2)\times(n^2)$	$A=\min(m,n)$
	$B=\min(m-a,n-a)$
	$C=\min(m-a-b,n-a-b)$
$(m^2)\times(2n)$	$A=\min([m/2],[n/2])$
	$B=\min([m/2]-a,n-2a)$
	$C=\min([m/2]-a-b,n-2a-b)$
	$D=\min(m-2a-2b-2c,n-2a-b-c)$
	$E=\min(m-2a-2b-2c-d,n-2a-b-c-d)$
	$F=\min(m-2a-2b-2c-d-e,n-2a-b-c-d-e)$

That these E_6 tensor product decomposition formulae (7.4) are all stable both with respect to m for fixed n and with respect to n for fixed m can be seen from the nature of the upper limits of summation appearing in table 30. In fact the expansions (7.4*a*)–(7.4*e*) are stable with respect to m for fixed n if $m\geqslant n, n, 2n, 2n, 2n$, respectively, and stable with respect to n for fixed m if $n\geqslant m, m, [m/2], [m/2], [m/2]$, respectively. The structures of the formulae were arrived at by using SCHUR to evaluate explicit products with large values of m and n. The generic formuale arising for sufficiently large m and sufficiently large n were themselves then checked dimensionally using Maple as polynomials in m for various fixed n, and then as polynomials in n for fixed m.

The other E_6 tensor products listed in table 29, namely $(m:m^5)\times(n:n^5)$, $(m:m^5)\times(2n:0)$, $(m:m^5)\times(2n:n^2)$ and $(m:m^5)\times(2n:n^4)$ may be obtained by taking the contragredient (7.4*a*), (7.4*c*), (7.4*e*) and (7.4*d*), respectively.

In the case of the case 3 m- and n-dependent E_7 tensor products we find

$$(m^2)\times(2n,n^6)=\sum_{a,b,c=0}^{A,B,C}(m+2n-2a-2b-c,m+n-2a-b-2c,n-b-c,(n-a-b-c)^4) \quad (7.5a)$$

$$(m^2)\times(n^2)=\sum_{a,b,c=0}^{A,B,C}(m+n-2a,m+n-2a-b-c,b+c,c^4) \quad (7.5b)$$

$$(m^2)\times(2n)=\sum_{a,b,c,d,e,f=0}^{A,B,C,D,E,F}(m+2n-2a-2b-c-d-e,m-a-2b-c,a+c+d+e+f,a+c+e+f,a+e+f,e+f,e) \quad (7.5c)$$

where the upper limits $A, B, \ldots, F$ of the summation parameters are given in table 31.

Once again, the fact that the E_7 tensor product decomposition formulae (7.4) are all stable both with respect to m for fixed n and with respect to n for fixed m can be seen from the nature of the upper limits of summation appearing in table 31. This time the expansions (7.4*a*)–(7.4*e*) are stable with respect to m for fixed n if $m\geqslant 2n, n, n$, respectively, and stable with respect to n for fixed m if $n\geqslant [m/2], m, m$, respectively. The structures of the formulae were again arrived at by using SCHUR to evaluate explicit products with large values of m and n. The generic formulae arising for sufficiently large m and sufficiently large n were themselves then checked dimensionally using Maple as polynomials in m for various fixed n, and then as polynomials in n for fixed m.

8. Tensor products of fundamental irreps with irreps of breadth > 1

In case 4 we consider tensor products of fundamental irreps $(\mu) = (\omega_i)$, for which $b(\mu) = w(\mu) = 1$, with irreps (ν) with $b(\nu) > 1$ but excluding the cases for which $(\omega_i) = (\omega)$, $(\bar{\omega})$ or (θ) where (ω) and (θ) are the defining and adjoint irreps, respectively. In testing for multiplicity-free products we may set aside those tensor products $(\omega_i) \times (\nu)$ for which $\nu = \sum_{j=1}^{k} a_j\omega_j$ has a non-vanishing component $a_j\omega_j$ in the fundamental basis such that $(\omega_i) \times (\omega_j)$ is not multiplicity free. Corollary 5 implies that such products cannot be multiplicity free. To proceed further one examines those cases of the form $(\omega_i) \times (\omega_j + \omega_k)$ with $j \neq k$. Considering each exceptional Lie group in turn one readily concludes that the only exceptional Lie group multiplicity-free products of the type $(\mu) \times (\nu)$ with $b(\mu) = w(\mu) = 1$ and $b(\nu) \geqslant 2$ are the following:

G_2: $(1) \times (\nu)$ for all (ν); (8.1a)

F_4: $(1) \times (\nu)$ for all (ν) with either $\nu_1 = \nu_2 + \nu_3 + \nu_4$ or $\nu_4 = 0$ or both; (8.1b)

E_6: $(1:1) \times (\nu)$ and $(1:1^5) \times (\nu)$ for all (ν); (8.1c)

E_7: $(1^2) \times (\nu)$ for all (ν). (8.1d)

These have all been dealt with previously since in each case (μ) is either the defining irrep (ω) or its contragredient $(\overline{\omega})$.

9. Tensor product of a breadth 1 irrep of Dynkin weight > 1 with an irrep of breadth > 1

In case 5 we consider products of the type $(\mu) \times (\nu) = (m\omega_i) \times (\nu)$ with $b(\mu) = 1$, $w(\mu) = m > 1$ and $b(\nu) \geqslant 2$.

By the usual argument based on corollary 5 we can confine attention to the multiplicity-free products appearing under case 4, with $(\mu = \omega_i)$ scaled by m with $m > 1$. The relevant (μ) are the natural irreps (ω) of G_2, F_4, E_6 and E_7, together with the irrep $(\overline{\omega})$ of E_6. For $(\nu) = (\omega_j + \omega_k)$ we need only consider those cases for which, in the tabulations of case 2, (ω_j) and (ω_k) specify distinct rows in which the entries in the column specified by $(n\omega)$ are 1, and neither $*$ nor $**$. In each case we start with a scaling factor $m = 2$.

Examining each exceptional Lie group in turn we find that the only multiplicity-free cases occur in E_6. These are the cases

$$(2\omega) \times (\omega + \overline{\omega}) = (2:2) \times (2:21^4) \qquad (9.1a)$$

$$(2\overline{\omega}) \times (\omega + \overline{\omega}) = (2:2^5) \times (2:21^4). \qquad (9.1b)$$

More generally, it has been found that the products

$$(m\omega) \times (n\omega + p\overline{\omega}) \qquad \text{and} \qquad (m\overline{\omega}) \times (n\omega + p\overline{\omega}) \qquad (9.2)$$

or, equivalently,

$$(m:m) \times (n+p:n+p, p^4) \qquad \text{and} \qquad (m:m^5) \times (n+p:n+p, p^4) \qquad (9.3)$$

are multiplicity free for all positive integer values of m, n and p. In particular we have the expansion

$$(m:m) \times (n+p:n+p, p^4) = \sum_{a,b,c,d,e=0}^{A,B,C,D,E} (m+n+p-a-2b-d : m+n+p$$
$$-2a-2b-2c-d-e, p-b-c+d+e, (p-b-c+d)^3) \qquad (9.4)$$

where the upper limits $A, B, \ldots, E$ of the summation parameters are given in table 32.

Table 32. Upper limits of summation for the E_6 tensor product expansion (9.4).

$(m : m) \times (n + p : n + p, p^4)$	$A = \min(p, n)$
	$B = \min(m - a, p - a)$
	$C = \min(m - a - b, p - a - b)$
	$D = \min(m - a - b - c, n - a)$
	$E = \min(m - a - b - c - d, n - a - d)$

The expansion of the E_6 tensor product $(m : m^5) \times (n + p : n + p, p^4)$ is just the contragredient of that given in (9.4).

It has been verified that the decomposition (9.4) is stable with respect to m for fixed n and p if $m \geqslant n + p$, stable with respect to n for fixed m and p if $n \geqslant m$ and stable with respect to p for fixed m and n if $p \geqslant n$. Each of the corresponding generic formulae has been checked dimensionally using Maple, expressing the left- and right-hand sides of (9.4) as polynomials in m, n and p, as appropriate.

10. Remaining cases

It only remains to consider those products $(\mu) \times (\nu)$ with $b(\mu) > 1$ and $b(\nu) > 1$ that constitute case 6. Using the above results on products with $b(\mu) = 1$ and $w(\mu) = w \geqslant 1$ and $b(\nu) > 1$, together with the usual weakly monotonic increasing argument, the only possible candidates are E_6 products of the form $(\mu) \times (\nu) = (m\omega + n\overline{\omega}) \times (p\omega + q\overline{\omega})$ with $m, n, p, q \geqslant 1$. However, in the case $m = n = p = q = 1$ we find the following maximum multiplicity:

		100 010 $(2 : 21^4)$
100 010	$(2 : 21^4)$	3

It follows that there are no multiplicity-free tensor products of the type $(\mu) \times (\nu)$ with $b(\mu) > 1$ and $b(\nu) > 1$ for E_6, nor indeed for any of the exceptional Lie groups.

11. Summary and conclusion

We can now collect together in one place the list of all multiplicity-free tensor products for each of the exceptional Lie groups as follows.

G_2: $(0) \times (\nu)$ and $(1) \times (\nu)$ for all (ν)
$(21) \times (n), (2n, n)$
$(2) \times (2n, n)$ (11.1)

F_4: $(0) \times (\nu)$ for all (ν)
$(1) \times (\nu)$ for all (ν) with either $\nu_1 = \nu_2 + \nu_3 + \nu_4$ or $\nu_4 = 0$ or both
$(1^2) \times (n), (n^2), (2n, n), (3n, n^3), (\Delta; 3n + 1, n^3)$
$(\Delta; 1) \times (n^2)$
$(2) \times (n^2)$ (11.2)

E_6: $(0 : 0) \times (\nu), (1 : 1) \times (\nu)$ and $(1 : 1^5) \times (\nu)$ for all (ν)
$(2 : 0) \times (n : n), (n : n^5), (2n : 0), (2n : n^2), (2n : n^4), (3n : n^3)$
$(2 : 1^2) \times (2n : 0)$ and $(2 : 1^4) \times (2n : 0)$
$(m : m) \times (n : n), (n : n^5), (2n : 0), (2n : n^2), (2n : n^4), (n + p : n + p, p^4)$

$$(m:m^5)\times(n:n),(n:n^5),(2n:0),(2n:n^2),(2n:n^4),(n+p:n+p,p^4) \tag{11.3}$$

E_7:

$$\begin{aligned}
&(0)\times(\nu) \quad \text{and} \quad (1^2)\times(\nu) \quad \text{for all } (\nu)\\
&(21^6)\times(2n),(n^2),(2n,n^2),(2n,n^6),(3n,n^3),(3n,n^5),(4n,n^5)\\
&(2)\times(2n),(n^2),(2n,n^6)\\
&(21^2)\times(n^2),(2n,n^6)\\
&(2^2)\times(2n,n^2)\\
&(m^2)\times(n^2),(2n),(2n,n^6)
\end{aligned} \tag{11.4}$$

E_8:

$$\begin{aligned}
&(0)\times(\nu) \quad \text{for all } (\nu)\\
&(21^7)\times(2n,n),(2n,n^7),(3n),(3n,n^6),(4n,n^2),(4n,n^5)\\
&(5n,n^4),(6n,n^4)\\
&(21)\times(2n,n),(2n,n^7).
\end{aligned} \tag{11.5}$$

In each case we have presented either in a table or the text an explicit formula for the decomposition of the corresponding tensor product which is manifestly multiplicity free. In the case of those parametrized by m, n or p the resulting formulae all show themselves to be stable with respect to sufficiently large values of these parameters, as required by proposition 1, and the resulting generic formulae have all been checked dimensionally for accuracy as polynomials in m, n or p. The formulae have all been expressed in terms of standard labels, but since every multiplicity-free product is of the form $m\omega_i \times (n\omega_{j_1} + p\omega_{j_2})$ with $1 \leqslant i, j_1, j_2 \leqslant k$ and $m, n, p \geqslant 0$ it is rather easy to read off the corresponding Dynkin labels $((a))$ for the relevant irreps from table 3.

Finally, we would like to point that proposition 1, that has proved to be so useful here in the multiplicity-free context, remains valid and of considerable use in the case of tensor products that are not multiplicity free. These, also, are subject to certain structural stability properties.

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